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Quark scattering amplitudes at strong coupling

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Abstract: Following Alday and Maldacena [1], we describe a string theory method to compute the strong coupling behavior of the scattering amplitudes of quarks and gluons in planar $\mathcal{N} = 2$ super Yang-Mills theory in the probe approximation. Explicit predictions for these quantities can be constructed using the all-orders planar gluon scattering amplitudes of $\mathcal{N} = 4$ super Yang-Mills due to Bern, Dixon and Smirnov [2].

Keywords: AdS-CFT Correspondence, Supersymmetric gauge theory, Strong Coupling Expansion.

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Contents

1. Introduction

A great deal of effort has been devoted to the computation of scattering amplitudes of massless partons in non-abelian gauge theories [3]. Such amplitudes are IR divergent and require a regulator, which is always present in a physical observable e.g. [4]. Via factorization theorems [5], these quantities are used to make predictions for high-energy QCD collision events. Amplitudes with both gluons and quarks are required for precise theoretical predications at high-energy colliders.

Some of the progress in constructing such amplitudes is facilitated by supersymmetry [3, 6]. S-matrix elements for massless partons in supersymmetric gauge theories are useful in a number of ways for physical high-energy QCD calculations. Firstly, perturbative SYM scattering amplitudes share many qualitative properties with QCD amplitudes in the regime relevant for jet physics. The Supersymmetric Yang-Mills (SYM) answers, however, are simpler than the QCD amplitudes, and hence are useful in learning to understand their structure. More pragmatically, tree level amplitudes with quarks can be related to supersymmetric amplitudes with gluinos using color manipulations; supersymmetry Ward identities then relate them to gluon amplitudes. SYM theories can be used as a testing ground for computational techniques. Finally, amplitudes in supersymmetric theories are useful as building blocks for those of QCD [6].

Work on planar gluon scattering in the $\mathcal{N} = 4$ theory has culminated in a conjecture by Bern, Dixon and Smirnov (BDS) [2] for the all-orders n-point planar MHV amplitudes. Very recently, Alday and Maldacena (AM) uncovered some of this structure in the string theory dual description [1]. In their ground-breaking paper, the authors show that the gauge theory S-matrix can be defined and regulated from the AdS string theory. The S-matrix is controlled at large 't Hooft coupling λ by a classical worldsheet. AM match the Sudakov IR divergence structure using local features of the extremal worldsheet. They also find the explicit worldsheet configuration for four gluons, using a solution of [7]. When combined with knowledge of the strong-coupling behavior of the cusp anomalous dimension [8, 7, 9], the area of the four-gluon worldsheet can be seen to match the strong coupling momentum-dependence predicted by the BDS ansatz.

Finding other explicit worldsheet solutions seems to be quite difficult. In order to understand the purview and utility of the AM procedure, it is worthwhile to try to extend it to other theories. The simplicity of the strong-coupling limit implied by the existence of a perturbative string theory description extends also to models with matter in the fundamental representation, such as those of [10, 11]. While tree level amplitudes with quarks can be reconstructed from gluon amplitudes as described above, this is no longer true at loop level, and new types of behavior can emerge. It would therefore be valuable to extend the strong-coupling description to such theories.

In this paper, we generalize the prescription of [1] to the holographic dual of a gauge theory with matter in the fundamental representation of the gauge group, which we will generically call quarks. The model we study arises by adding a small number of D7-branes to $AdS_5 \times S^5$ in a way that preserves eight supercharges [10, 11]. At strong coupling, scattering amplitudes in the theory with flavor are governed a simple generalization of the minimal-surface problem studied by AM.

Our most interesting result is a relation between field theories. From a symmetry relation between their strong-coupling dual worldsheets, we will find a set of equations relating gluon scattering in the $\mathcal{N} = 4$ theory and quark-and-gluon scattering in the flavored theory. We can then exploit the conjecture of BDS [2] to make predictions for quark scattering amplitudes.

The paper is organized as follows. In section two and appendix A, we review the conjecture of BDS for planar gluon scattering. In section three, we give a detailed exposition

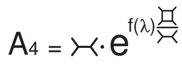

Figure 1: The BDS ansatz for the four gluon amplitude, where $f(\lambda)$ is some function of the coupling.

of the prescription of Alday and Maldacena. In the course of reviewing the method, we explain how the positions of the vertex operators are determined by the T-dual solution.[1] In section four, we explain the generalization of the prescription to include fields in the fundamental representation of the gauge group. Section five describes a relationship between quark and gluon scattering which follows from their string theory descriptions. Using this relationship and the BDS ansatz, we make an explicit prediction for the $\bar{q}ggq$ amplitude. In section six, we study the IR divergences of the resulting quark scattering amplitudes, and verify the divergent part of the result with a direct argument in the gauge theory with fundamentals. In section seven we discuss the comparison of our prediction with perturbative $\mathcal{N}=2$ results. Toward that aim, we also study the Regge limit of quark scattering amplitudes in $\mathcal{N}=2$ SYM.

2. Gluon scattering in planar $\mathcal{N}=4$ SYM

Based on explicit computation of the planar four-gluon scattering amplitude up to four loops, Bern, Dixon and Smirnov (BDS) gave an ansatz for the all-loop planar n-gluon amplitudes $\mathcal{A}_n$ in $\mathcal{N}=4$ SYM (at least, for the maximally helicity violating ones) [2]. At strong coupling, Alday and Maldacena (AM) have confirmed the BDS ansatz for the 4-gluon amplitude (for the $in \to out \to in \to out$ momentum ordering) using the AdS/CFT correspondence [1]. The form of the BDS ansatz for the four gluon amplitude can be described graphically as in figure 1.

More precisely, the gluon amplitude is IR divergent and needs an infrared regulator. In dimensional regularization $d = 4 - 2\epsilon$,[2] the BDS ansatz for the n-gluon amplitude is

$$\mathcal{A}_n = \mathcal{A}_n^{\text{tree}} \, e^{-\mathcal{S}_n} \, , \tag{2.1}$$

where $\mathcal{A}_n^{\text{tree}}$ is the tree-level amplitude and

$$-\mathcal{S}_n = \sum_{l=1}^{\infty} \lambda_\epsilon^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \tag{2.2}$$

does not depend on any color or helicity factors. The symbols appearing in (2.2) are defined as follows. $M_n^{(1)} = \mathcal{A}_n^{(1)}/\mathcal{A}_n^{\text{tree}}$ is the ratio of one-loop and tree amplitudes.

$$\lambda_\epsilon = \frac{g^2 N}{8\pi^2} \left(\frac{4\pi}{\mu^2 e^\gamma} \right)^\epsilon , \qquad \gamma = -\Gamma'(1) \tag{2.3}$$

[1] We thank Hong Liu for asking this question.

[2] More precisely, in *four-dimensional-helicity* scheme, in which all helicity states are four-dimensional and only the loop momentum is continued to $d = 4 - 2\epsilon$ dimensions.

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is the 't Hooft coupling,[3] and μ is an IR cutoff. $f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\epsilon + f_2^{(l)}\epsilon^2$ is a set of functions, one at each loop order, which make their appearance in the exponentiated all-loop expression for the infrared divergences in generic amplitudes [12]. In particular, $f(\lambda) \equiv 4\sum_l f_0^{(l)}\lambda^l$ is the cusp anomalous dimension (equal to the anomalous dimension of twist-two operators of large spin). Its large 't Hooft coupling asymptotic is $f(\lambda) \sim \sqrt{\lambda}+\text{const}+O(1/\sqrt{\lambda})$ [8, 7, 9]. An important aspect of the conjecture is that the constants $C^{(l)}$ do not depend on kinematics or on the number of particles n. The non-iterating remainders $E_n^{(l)}$ vanish as $\epsilon \to 0$ and depend explicitly on n.

$\mathcal{S}_n$ can be decomposed into finite and divergent pieces as $\mathcal{S}_n = \mathcal{S}_n^{\text{div}} + \mathcal{S}_n^{\text{fin}}$, where

$$-\mathcal{S}_n^{\text{div}} = \sum_{l=1}^{\infty} \lambda_\epsilon^l f^{(l)}(\epsilon) I_n^{(1)}(l\epsilon) \tag{2.4}$$

with (the 1-loop scalar box integral)

$$I_n^{(1)}(\epsilon) = -\frac{1}{2}\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon}. \tag{2.5}$$

$s_{i,i+1}$ is *minus* the square of the sum of the i and $i+1$ external (outgoing) momenta, and as apposed to the convention used in [2], here we work with $(-+++)$ signature. $f_1^{(l)} \equiv \frac{l}{2}\mathcal{G}_0^{(l)}$ is the sub-leading divergent term. Expanding (2.4) in powers of ϵ, in the limit $\epsilon \to 0$ it gives[4]

$$-\mathcal{S}_n^{\text{div}} = -\frac{f(\lambda)}{16}\ln^2\left(\frac{\mu^2}{-s_{i,i+1}}\right) - \frac{g(\lambda)}{4}\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right) - \frac{1}{2}h(\lambda)\,, \tag{2.6}$$

where $g(\lambda) = \sum \lambda^l \mathcal{G}_0^{(l)}$ and $h(\lambda) = \sum \lambda^l f_2^{(l)}/l^2$.[5]

The finite part can be written in terms of its 1-loop counterpart $F_n^{(1)}$ as:

$$-\mathcal{S}_n^{\text{fin}} = \frac{f(\lambda)}{4}F_n^{(1)} + C(\lambda)\,, \tag{2.7}$$

where $C(\lambda)$ depends neither on n, nor on momenta. The one-loop finite remainder, $F_n^{(1)}$ was evaluated in [13]. Since in sections 5 and 7 we will need its explicit value, we write it in appendix A.

2.0.1 Double-loop representation of the one-loop amplitude

As was recently shown in [14, 15] the one-loop n-gluon amplitude, $M_1^{(n)}$ (which contains both the divergent and finite pieces), can be written as a simple double contour integral. This expression has a nice geometrical interpretation as the dimensionally regularized, one-loop contribution to a polygonal Wilson loop Π, made from the successive external

[3]Note that for $\epsilon \neq 0$ the gauge coupling g is dimensionful, whereas λ_ϵ is dimensionless.

[4]The $1/\epsilon^2$ and $1/\epsilon$ terms cancel in physical processes, as per the theorems of Bloch-Nordsieck and Kinoshita-Lee-Nauenberg, e.g. [4].

[5]Note that g and h change if we rescale the IR cutoff.

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momenta (see section 3):[6]

$$M_1^{(n)} = \frac{1}{2}\mu^{2\epsilon} \oint_\Pi \oint_\Pi \frac{d\mathbf{y}\cdot d\mathbf{y}'}{[-(\mathbf{y}-\mathbf{y}')^2]^{1+\epsilon}} \ . \tag{2.8}$$

In [1], in addition to dimensional regularization, another regularization that is more natural from the AdS/CFT point of view was used. In AdS, it corresponds to cutting the Poincaré radial coordinate at some small value $r_{\rm IR} = 1/z_{\rm IR}$ and imposing the boundary conditions there. The double loop integral (2.8), has a natural adaptation to the AdS-regularization:

$$\widetilde{M}_1^{(n)} = \frac{1}{2} \oint_\Pi \oint_\Pi \frac{d\mathbf{y}\cdot d\mathbf{y}'}{(\mathbf{y}-\mathbf{y}')^2 + 1/z_{\rm IR}^2} \ . \tag{2.9}$$

However, we have not explicitly compared this expression with the area of the AM 4-gluon solution with a radial cutoff. Note that although the boundaries of the polygon loop are still null (and are therefore T-dual to null gluons), the particle being exchanged is now massive.

In this paper we will *assume* that the BDS ansatz (2.1) is correct and will combine it with its AdS dual picture to give a prediction for the quark and gluon-quark planar amplitude at strong coupling.

3. The strong coupling dual of planar $\mathcal{N}=4$ SYM gluon scattering

In [1], Alday and Maldacena used the AdS/CFT correspondence to compute gluon amplitudes in the t' Hooft limit of N = 4 SYM at strong coupling.

On the string theory side, the leading order result at strong coupling is given by a single classical string configuration associated to the scattering process. As explained in [1], the string theory scattering in AdS is happening at fixed angles and large proper momentum and it is thus determined by a classical solution. The final form for the color ordered planar scattering amplitude of n gluons at strong coupling is of the form

$$\mathcal{A} \sim e^{-S_{cl}} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})_{cl}} \ , \tag{3.1}$$

where S_{cl} denotes the action of a classical solution of the string worldsheet equations of motion, which is proportional to the area of the string world-sheet and $\sqrt{\lambda} = R^2_{\rm AdS}/l_s^2$.[7] The solution depends on the 4-momenta, $\mathbf{k}_i$, of the gluons. The whole dependence on the coupling is in the overall factor.

In more detail, one has to compute the worldsheet path integral over worldsheets with the topology of a disk, embedded in Poincaré AdS:

$$ds^2 = R^2_{AdS}\frac{dz^2 + dx^2_{3+1}}{z^2} \ . \tag{3.2}$$

[6] Note the sign difference in the dimensional reduced Wilson loop computation at $d = 4-2\tilde{\epsilon}$, $\tilde{\epsilon} = -\epsilon > 0$. This can be understood as a consequence of the fact that T-duality interchanges the UV with the IR and therefore the sign of ϵ. This surprising relationship between field theory quantities is made somewhat less shocking in light of the AM prescription, reviewed in the next section. We record it here because it will be used in section 6. The relationship between the scattering amplitude and the Wilson loop has recently been confirmed in more detail in [16].

[7] We set $\alpha' = 1$.

The open strings have Neumann boundary conditions in the 3+1 spatial directions and fixed radial position z_{IR} at the worldsheet boundary (i.e. Dirichlet boundary condition in z). In addition one should add open string insertions, dual to the asymptotic gluons being scattered. These vertex operators have fixed dual field theory 4-momentum $\mathbf{k}$ along the spatial directions $\mathbf{x}$. At z_{IR}, the proper string momentum (conjugate to the coordinates $d\hat{\mathbf{x}} = \lambda^{1/4} d\mathbf{x}/z_{IR}$) is $\mathbf{k}\, z_{IR}/\lambda^{1/4}$ and therefore, in a limit where $z_{IR} \to \infty$ faster than $\lambda^{1/4}$, the proper momentum diverges. The problem becomes a classical problem of finding the saddle point [17] of the worldsheet path integral. We will discuss the validity of this saddle-point approximation at the end of §3.2.

3.1 Using T-duality to simplify the boundary conditions

It will be useful to review the solution of this steepest-descent problem in some detail. The open string vertex operators are

$$V_{open}(\mathbf{k}_i; \sigma_i) \propto e^{i\mathbf{k}_i \cdot \mathbf{x}(\sigma_i)}\,, \tag{3.3}$$

where $\sigma \in [-\infty, \infty)$ parameterizes the boundary of the disk (upper half plane), $\mathbf{x}(\sigma) = \mathbf{x}(\sigma, \tau)|_{\tau=0}$ and $\mathbf{k}^2 = 0$. The vertex operators can be rewritten as contributions to the boundary action as:

$$\begin{aligned} i\sum_{j=1}^{n} \mathbf{k}_j \cdot \mathbf{x}(\sigma_j) &= i\sum_{j=1}^{n} \mathbf{k}_j \cdot \int d\sigma\ \mathbf{x}(\sigma)\delta(\sigma - \sigma_j) = i\sum_{j=1}^{n-1} \mathbf{k}_j \cdot \int d\sigma \mathbf{x}(\sigma)\partial_\sigma \theta(\sigma; \sigma_j, \sigma_n) \\ &= -i\sum_{j=1}^{n-1} \mathbf{k}_j \cdot \int d\sigma \partial_\sigma \mathbf{x}(\sigma)[\theta(\sigma;\sigma_j,\sigma_n) + c] \\ &= -i\sum_{j=1}^{n} \int_{\sigma_j}^{\sigma_{j+1}} d\sigma \partial_\sigma \mathbf{x}(\sigma) \cdot \left(\sum_{i\le j} \mathbf{k}_i + \mathbf{c}\right)\,, \end{aligned} \tag{3.4}$$

where θ is the periodic step function

$$\theta(\sigma; \sigma_i, \sigma_j) = \begin{cases} 1 & \sigma_i < \sigma < \sigma_j \\ 0 & \text{otherwise} \end{cases}\,, \tag{3.5}$$

$\sigma_1 < \sigma_2 < \ldots < \sigma_n$, $\sigma_{n+1} = \sigma_1$ and $\mathbf{c}$ is a constant 4-vector.[8]

Next [1], as a technical trick to find the saddle point, we do a change of variables in the path integral which can be described as a "T-duality" along the non-compact 3+1 flat directions. To do this, we follow Buscher [18]. For each field x^μ, we gauge the shift symmetry $x^\mu \to x^\mu + a$, and introduce a worldsheet gauge field A^μ_α and a scalar lagrange multiplier y^μ. We then consider the gauge-invariant action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma\partial\tau\ \left[(\partial_\alpha \mathbf{x} - \mathbf{A}_\alpha)^2/z^2 - i\mathbf{y}\cdot\mathbf{F}\right] + i\sum_{j=1}^{n-1} \int_{\sigma_j}^{\sigma_{j+1}} d\sigma [\partial_\sigma \mathbf{x} - \mathbf{A}_\sigma] \cdot \left(\sum_{i\le j} \mathbf{k}_i + \mathbf{c}\right)\,, \tag{3.6}$$

[8]Note that since $\sum_{j=1}^{n-1} \mathbf{k}_j = -\mathbf{k}_n$, the sum in the second line of (3.4) runs up to $n-1$ and we could equivalently choose to omit any other $\mathbf{k}_{j<n}$ instead of $\mathbf{k}_n$.

where $\mathbf{F} = \partial_\tau \mathbf{A}_\sigma - \partial_\sigma \mathbf{A}_\tau$, $(\partial_\tau \mathbf{x} - \mathbf{A}_\tau)\,|_{\tau=0} = 0$ and we are suppressing the kinetic term for z. Now we can gauge fix $\mathbf{x} = 0$, so the action becomes

$$S = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma \partial\tau \ \left[\mathbf{A}_\alpha \cdot \mathbf{A}_\alpha / z^2 - i \mathbf{y} \cdot \mathbf{F}\right] - i \sum_{j=1}^{n-1} \int_{\sigma_j}^{\sigma_{j+1}} d\sigma \mathbf{A}_\sigma \cdot \left(\sum_{i \leq j} \mathbf{k}_i + \mathbf{c} \right) . \tag{3.7}$$

If we first integrate out $\mathbf{y}$, then we see that $A_\alpha = -\partial_\alpha \widetilde{\mathbf{x}}$ is a flat connection and therefore (3.6) is equivalent to the original action. If on the other hand, we first integrate $\mathbf{A}$, then it is convenient to integrate by parts in the second term in (3.7). We then have

$$\begin{aligned} S = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} & d\sigma \partial\tau \ \left[\mathbf{A}_\alpha \cdot \mathbf{A}_\alpha / z^2 + i \left(\mathbf{A}_\sigma \cdot \partial_\tau \mathbf{y} - \mathbf{A}_\tau \cdot \partial_\sigma \mathbf{y}\right)\right] \\ & - i \sum_{j=1}^{n} \int_{\sigma_j}^{\sigma_{j+1}} d\sigma \mathbf{A}_\sigma \cdot \left(\sum_{i \leq j} \mathbf{k}_i + \mathbf{c} + \frac{\sqrt{\lambda}}{4\pi} \mathbf{y} \right) . \end{aligned} \tag{3.8}$$

It is convenient to rescale $\mathbf{A}$, $\mathbf{y}$ and z as $(\mathbf{A}, \mathbf{y}, z) \to \left(\frac{\sqrt{\lambda}}{4\pi}\mathbf{A}, \frac{4\pi}{\sqrt{\lambda}}\mathbf{y}, \frac{\sqrt{\lambda}}{4\pi} z\right)$, so that $\sqrt{\lambda}$ will stand in front of the whole action. Integrating out $\mathbf{A}$ we find in the bulk of the disk an action for $(\mathbf{y},\ r = 1/z)$ describing a dual AdS background[9]

$$ds^2 = R^2_{AdS} \frac{dr^2 + d\mathbf{y} \cdot d\mathbf{y}}{r^2} . \tag{3.9}$$

Integrating over the boundary value of $\mathbf{A}_\sigma$ enforces the boundary condition

$$\mathbf{y}(\sigma_i < \sigma < \sigma_{i+1}) = -\sum_{j \leq i} \mathbf{k}_j - \mathbf{c} . \tag{3.10}$$

The boundary condition (3.10) means that $\mathbf{y}$ is constant on any segment $\sigma_i < \sigma < \sigma_{i+1}$ and jumps by $-\mathbf{k}_i$ at σ_i. Due to the momentum conservation $\sum_i \mathbf{k}_i = 0$, the boundary of the string in the T-dual $\mathbf{y}$-coordinates is closed. It stretches along a polygonal loop made of the ordered open string momenta $\mathbf{k}_i$, centered at $\mathbf{c}$. $\mathbf{c}$ drops out of the calculation and will be set to zero from now on.

In addition, the Gaussian integration over $\mathbf{A}$ introduces a linear dilaton ($\Phi \sim \log(z)$). It can be ignored in the classical saddle point approximation because the dilaton term does not scale as $\sqrt{\lambda} = R^2_{AdS}/l_s^2$; so as long as $\sqrt{\lambda} \gg \log(z_{IR})$, it can be neglected with respect to the kinetic and boundary terms.

Note that we have "T-dualized" along non-compact directions as well as time. That was possible because we were interested only in the classical limit of the disk amplitude, as opposed to a quantum mechanical object which would involve winding of closed strings. The non-compact "T-duality" may not be valid if one considers worldsheets with non-trivial topology.

[9]Note that r and $\mathbf{y}$ have dimensions of $\frac{1}{\text{length}}$. Note that in terms of the original un-rescaled radial coordinate z, we have $r = \sqrt{\lambda}/z$.

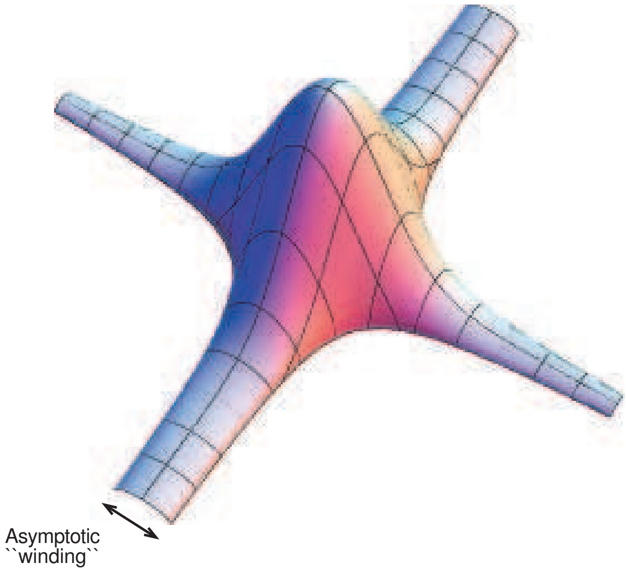

Figure 2: A rough picture of the 4-gluon worldsheet embedding in AdS_3. The worldsheet has a Dirichlet boundary condition in the radial direction and may have asymptotic extent ("winding") in the other directions, as indicated. The extremum of the path integral over the vertex operator insertion points corresponds to the embedding where the asymptotic string states have zero extent. It is strongly peaked as the proper momentum $kz_{IR} \to \infty$.

3.2 The role of the vertex insertion points

The solution to the equation of motion is an extremal-area surface in the T-dual $\mathbf{y}$-coordinates subject to the boundary conditions (3.10). For any distribution of the ordered open string vertex operators $\sigma_1 < \sigma_2 < \ldots < \sigma_n$, there is such a solution, whose fluctuations are suppressed at large λ. Physically, these extremal surfaces solution differ by the momentum flow along the $\mathbf{y}$-segments which is controlled by the values of the σ's. After changing variables back to the $\mathbf{x}$ space, this momentum flow translates into the extension ("winding") of the asymptotic classical open string state. If the momenta $\mathbf{k}_i$ are null, then, as we will next prove, the solution with the extremal surface area is the one which satisfies Neumann boundary condition in the $\mathbf{k}_i$ directions along the corresponding $\mathbf{y}$ segment. Therefore, for generic values of the σ's, the asymptotic open string states have non-zero extent (in the $\mathbf{x}$ coordinates).[10] At the extremum they have zero extent (zero "winding") as one would physically expect (see figure 1).

To see this explicitly, consider first the Gross-Manes flat space case [17, 20]. Their solution is

$$\mathbf{x} = -i \sum_j \mathbf{k}_j G(z, \sigma_i)\,, \tag{3.11}$$

[10] See [19] for such explicit solutions to the equations of motion, which end on the same polygonal locus but have nonzero momentum flow through the boundary. [19] show explicitly that the AM solution extremizes the area in this class of solutions.

where

$$G(z,\sigma) = \log |z-\sigma|^2 \tag{3.12}$$

is the Green function in two dimensions and we have parameterized the disk by the upper half plane $z = \sigma + i\tau$. "T-dualizing" (3.11), we get

$$\mathbf{y} = -i \sum_j \mathbf{k}_j \widetilde{G}(z,\sigma_i)\,, \tag{3.13}$$

where

$$\widetilde{G}(z,\sigma) = \log\left(\frac{z-\sigma}{\bar{z}-\sigma}\right) \tag{3.14}$$

is the T-dual Green function. The two Green functions are related by

$$\partial_\sigma G = (\partial + \bar\partial) G = (\partial - \bar\partial)\widetilde{G} = -i\partial_\tau \widetilde{G}\ . \tag{3.15}$$

Now, the action evaluated on a solution to the equation of motion is just a boundary term (when all the vertex operators are inserted on the boundary):

$$S = \frac{1}{2}\sum_{i\neq j} \mathbf{k}_i \cdot \mathbf{k}_j G(\sigma_i,\sigma_j)\ . \tag{3.16}$$

At the extremum, the variation of the action with respect to any of the open string vertex insertions (σ_i) must vanish and is given by

$$\begin{aligned}
\partial_{\sigma_i} S &= \frac{1}{2}\mathbf{k}_i \cdot \sum_j \mathbf{k}_j \partial_{\sigma_i} G(\sigma_j,\sigma_i) \\
&= -\frac{i}{2}\mathbf{k}_i \cdot \sum_j \mathbf{k}_j \partial_\tau \widetilde{G}(\sigma_j, z)|_{z=\sigma_i} \\
&= \frac{1}{2}\mathbf{k}_i \cdot \partial_\tau \mathbf{y}(z)|_{z=\sigma_i} = 0\,,
\end{aligned} \tag{3.17}$$

where we have used the null relation $\mathbf{k}_i \cdot \mathbf{k}_i = 0$. Equation (3.17) is the Neumann boundary condition on the corresponding $\mathbf{y}$ segment.[11]

Next, we give a general proof that extremizing the positions of boundary insertions implies the Neumann boundary condition. Unlike the previous one, this proof does not rely on having a flat background, and therefore applies in the AdS case as well. First note that

$$\partial_{\sigma_i} S_{\rm on-shell} = 0 \quad \Longleftrightarrow \quad \partial_{\sigma_i} e^{-S_{\rm on-shell}} = 0\ . \tag{3.18}$$

We are considering only the case where the saddle point approximation is valid. In such case, to leading approximation and for any values of the σ's, the amplitude is given by the exponential of the action, evaluated at the saddle point:

$$\langle V_1(\mathbf{k}_1,\sigma_1) V_2(\mathbf{k}_2,\sigma_2)\ldots V_n(\mathbf{k}_n,\sigma_n)\rangle = e^{-S_{\rm on-shell}}\ . \tag{3.19}$$

[11] Note that in these singular coordinates, any of the $\mathbf{y}$ segments is mapped into a single worldsheet point σ_i.

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Now, rewrite the left hand side of (3.19) as a path integral with the action (3.8), which we label as $\widetilde{S}$. That is, the path integral, labeled as $\widetilde{Z}$, is also over the gauge field $\mathbf{A}$ and $\mathbf{y}$. At any point in the functional integration, the fields are integration variables and so their values are some smooth functions, independent of σ_i. In that representation (see (3.8)),

$$\partial_{\sigma_i} e^{-\widetilde{S}} = i\mathbf{k}_i \cdot \mathbf{A}_\sigma(\sigma_i) e^{-\widetilde{S}} \ . \tag{3.20}$$

Plugging this relation into the new path integral (where (3.8) is the action) and integrating out the gauge field first, we get

$$0 = \partial_{\sigma_i} \log \widetilde{Z} = i\frac{\sqrt{\lambda}}{\widetilde{Z}} \langle\langle \mathbf{k}_i \cdot \mathbf{A}_\sigma(\sigma_i)\rangle\rangle = -\frac{z_{IR}^2}{\sqrt{\lambda}} \mathbf{k}_i \cdot \partial_\tau \mathbf{y}(\sigma_i) \quad \Longrightarrow \quad \mathbf{k}_i \cdot \partial_\tau \mathbf{y}(\sigma_i) = 0 \ . \tag{3.21}$$

The double angle braces $\langle\langle \ldots \rangle\rangle$ refer to averages over $\mathbf{A}$ but not $\mathbf{y}$. If, on the other hand we integrate out $\mathbf{y}$ first, then we get

$$0 = \partial_{\sigma_i} \log \widetilde{Z} = i\frac{\sqrt{\lambda}}{\widetilde{Z}} \langle\langle \mathbf{k}_i \cdot \mathbf{A}_\sigma(\sigma_i)\rangle\rangle = i\mathbf{k}_i \cdot \partial_\sigma \mathbf{x}(\sigma_i) \quad \Longrightarrow \quad \mathbf{k}_i \cdot \partial_\sigma \mathbf{x}(\sigma_i) = 0 \,, \tag{3.22}$$

which means that the asymptotic open string state created by V_i has zero "winding". It is important to note that (3.21) and (3.22) are evaluated on the saddle point of the original path integral

$$\widetilde{Z} = \langle V_1(\mathbf{k}_1, \sigma_1) V_2(\mathbf{k}_2, \sigma_2) \ldots V_n(\mathbf{k}_n, \sigma_n)\rangle \tag{3.23}$$

and not of the path integral

$$\langle\langle \mathbf{k}_i \cdot \mathbf{A}_\sigma(\sigma_i)\rangle\rangle = \langle V_1(\mathbf{k}_1, \sigma_1) V_2(\mathbf{k}_2, \sigma_2) \ldots \partial V(\mathbf{k}_i, \sigma_i) \ldots V_n(\mathbf{k}_n, \sigma_n)\rangle \,, \tag{3.24}$$

for which the saddle point in question does not contribute. The condition for extremizing over the positions of the vertex insertions is the statement that the descendant field $\mathbf{k} \cdot \partial\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}}$ decouples.

The width of the saddle point is given by

$$\begin{aligned} -\partial_{\sigma_j}\partial_{\sigma_i} S_{\text{on-shell}} &= i\frac{\sqrt{\lambda}}{\widetilde{Z}} \langle\langle \mathbf{k}_i \cdot \partial_{\sigma_j} \mathbf{A}_\sigma(\sigma_i)\rangle\rangle \\ &= i\delta_{ji}\frac{\sqrt{\lambda}}{\widetilde{Z}} \langle\langle \mathbf{k}_i \cdot \partial_\sigma \mathbf{A}_\sigma(\sigma_i)\rangle\rangle = -\delta_{ji}\frac{z_{IR}^2}{\sqrt{\lambda}} \mathbf{k}_i \cdot \partial_\sigma\partial_\tau \mathbf{y}(\sigma_i) \ . \end{aligned} \tag{3.25}$$

The leading (divergent) term in $\partial_\sigma\partial_\tau\mathbf{y}(\sigma_i)$ is proportional to $\mathbf{k}_i$ and therefore does not contribute to (3.25). Next, we note that the problem has a (dual conformal) symmetry under which $\mathbf{y}$ and $\mathbf{k}$ have charge $+1$, whereas z and z_{IR} have charge -1 (so (3.25) is invariant). Therefore

$$\partial_\sigma\partial_\tau\mathbf{y}(\sigma_i) = \sum_l \mathbf{k}_l \, \wp_{li}\left(\frac{z_{IR}^2 \mathbf{k}_m \cdot \mathbf{k}_n}{\lambda}\right) \,, \tag{3.26}$$

where $\wp_{li}$ is some (unknown) function that depends on the cross-ratio of vertex operator positions.

The inverse width of the saddle is

$$S'' \sim \frac{z_{IR}^2 \mathbf{k}_m \cdot \mathbf{k}_n}{\sqrt{\lambda}} \wp_{ij} \left(\frac{z_{IR}^2 \mathbf{k}_m \cdot \mathbf{k}_n}{\lambda} \right) . \tag{3.27}$$

In order to evaluate $\wp$, one needs some approximate solution near the polygon boundaries.[12] We expect (3.27) to be large in the regime of interest $\lambda \to \infty, z_{IR} \to \infty$, with

$$1 \ll \lambda^{1/2} \ll z_{IR}^2 \mathbf{k}_i \cdot \mathbf{k}_j \ll e^{2\sqrt{\lambda}} \qquad ; \tag{3.28}$$

the first limit is for the semiclassical approximation to the $\mathbf{x}, z$ path integral, the last is to avoid the large dilaton. Finally, the indicated behavior of z_{IR} is the only reasonable regime to study λ^{-1} corrections or to compare with weak-coupling results.

3.3 Extremum vs. minimum

Note that although both the Gross-Manes [20] and the Alday-Maldacena (AM) [1] solutions are extrema of the worldsheet area in the T-dual coordinates, they are not necessarily *minimal* area solutions (not even locally). For the Gross-Manes four-open-strings amplitude, this can be seen for example by noting that the second derivative of the on-shell action with respect to the cross ratio of the vertex insertion point can have different signs in different physical processes. In the AdS case, this can be seen by starting with the AM solution for four gluons and rescaling the radial coordinate (r in (3.9)), such that the new embedding extends more deeply into the AdS. The parts of the string worldsheet near the boundary now become more null and the part where it closes is pushed farther into the bulk; therefore the area decreases. This situation is familiar when one uses steepest descent to approximate a one-dimensional integral in the complex plane. In that case, there is always one direction along which the saddle point is a minimum and one along which it is a maximum.

3.4 Color ordering

For any given set of $n > 3$ gluon external momenta there can be $\frac{(n-1)!}{2}$ different orderings that are not related by cyclic permutation or reflection. In AdS, any of these orderings correspond to a different null polygon boundary condition and therefore to a different extremal area problem. In addition, the BDS ansatz applies for any of these orderings.[13] These facts suggest that we should expect to find corresponding $\frac{(n-1)!}{2}$ different extremal surfaces. We believe that this is indeed the case, but do not have a proof.

On the other hand, in the flat space case [20], one can do the same T-duality transformation and obtain a different extremal area problem for any ordering. However, in that case, by solving equation (3.17) for the four point amplitude one finds that there is an extremum only for spacelike s and t channel momentum transfer (only for the

[12] For the four gluon amplitude, one may evaluate $\wp$ by studying more carefully the explicit solutions in [1, 19].

[13] In AdS, reversing the ordering of the null segments is obtained by reflecting all the transverse directions, which is a symmetry. In the field theory, it is the result of charge conjugation (which may change the overall sign).

in $\rightarrow$ *out* $\rightarrow$ *in* $\rightarrow$ *out* momentum ordering) which, up to cyclic ordering and reflection, counts as one color ordering. That is also the only 4-gluon ordering for which we have a known AdS solution [1]. From the worldsheet point of view, the Chan-Paton factors generically restrict to a specific vertex operator ordering, which otherwise would all be part of the same moduli space. Therefore, one then would be tempted to infer that also in AdS that should be the case. That is, that for any set of external momenta, up to cyclic permutations and reflection of the ordering, there is a unique order for which the string theory path integral in AdS has an extremum.

As stated above, we believe that the last statement of the previous paragraph is false, and in this manner, AdS is different from flat space. That is, in AdS we expect a solution for any ordering. That point will be important when, in section 5, we will relate the quark amplitudes to the pure gluon ones.

3.5 Form factors

The elaboration of the T-duality transformation in this section has a possible generalization to the calculation of 'form factors' for scattering of gluons off of gauge-invariant operators dual to closed strings.[14] The closed string with nonzero momentum will create a cut on the T-dual worldsheet and prevents the boundary from forming a closed loop. One can envision a useful description along these lines in terms of the universal cover of the embedded string. In the special case that the closed-string vertex is on-shell, $0 = \mathbf{k}^2 + m^2 = q^2$ (so that it has no momentum in the AdS radial direction), it is pushed (in the classical solution) to the boundary of the worldsheet, and can be treated classically as an open-string insertion which does not participate in the Chan-Paton trace. (This is also what happens in the flat space Gross-Mende calculation with open strings and a single closed string insertion.) In such a case one finds the interesting problem of minimizing the action over the order in which the closed-string insertion appears.

4. The strong coupling dual of planar $\mathcal{N} = 2$ SYM quark scattering

We add to the $\mathcal{N} = 4$ theory N_f (in general, massive) $\mathcal{N} = 2$-preserving fundamental fields. At finite N_f/N, these break the conformal symmetry and the resulting theory needs a UV completion. In the gravity dual, this is described [11] by adding N_f D7-branes that wrap an $S^3 \subset S^5$ in the $AdS_5 \times S^5$ geometry. As a result the dual geometry is no longer asymptotically AdS. However, as long as $N_f \ll N_c = N$, we can ignore the back-reaction of the D7-branes and treat them as probes. This is dual to the statement that the field theory looks conformal, up to a very small scale. The addition of D7-branes introduces two new open string sectors: the 3-7 strings which are dual to fundamental hypermultiplet fields, and the 7-7 strings which are dual to operators of the form $\bar{q} \dots q$. The mass of the hypermultiplets is related to the extent of the 7-branes in the radial direction (see for example [22]). In this paper we will consider the case of massless quarks. Therefore, the D7-branes "wrap" the whole AdS_5.

[14] The possibility of studying such observables in the strong coupling description was raised in [21].

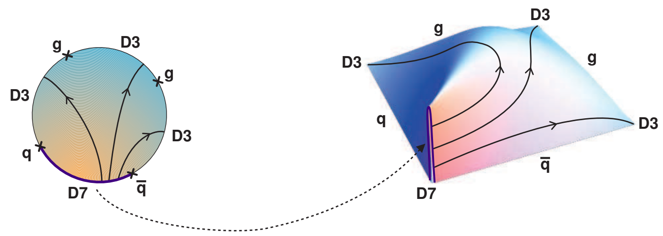


Figure 3: The worldsheet for $\bar{q}ggq$ scattering, and its image in the T-dual AdS. The quark and antiquark vertex operators change boundary conditions from the 3-branes to the 7-branes and back.

It is useful to think of the AM prescription as arising from the near-horizon limit of a flat-space open-string calculation. By adding D7-branes to this picture, amplitudes with quarks will arise from the near-horizon limit of disk amplitudes with vertex operators for 3-7 strings inserted on the boundary. Asymptotic (massless) 3-7 open strings do not extend into the radial AdS direction (or in any other direction). Therefore, if we fix their dual field theory momenta then, just as for gluons, their proper momentum is large and the saddle point approximation will be well-peaked.

Such amplitudes of massless partons are IR divergent and need regularization. In [1], two different regularizations were used. One regularization that is natural from the geometric AdS point of view, was to hold the branes associated with the color indices of the scattered gluons at some fixed radial position z_{IR}. This provides an IR cutoff on the calculation. This cutoff also regulates the quark amplitudes (as will soon be obvious). A second regularization used in [1] is dimensional regularization. It is achieved by replacing the D3-branes by Dp-branes, where $p = 3-2\epsilon$. It could be extended to the case at hand by replacing the D7-branes with D$\tilde{p}$-branes, where $\tilde{p} = 6-2\epsilon$; we do not pursue this possibility here.

4.1 Boundary conditions at the D7 cusp

To find the saddle point solution, it will again be useful to study the 'T-dual' description. The T-duality is done only along directions shared by the D3-branes and D7-branes. Each of the vertex operators has a $e^{i\mathbf{k}\cdot\mathbf{x}}$ factor, which causes the T-dual $\mathbf{y}$ coordinate to jump by $\mathbf{k}$ at the insertion point. Along the components of the boundary between the vertex operators, the image in the target space lies at a fixed value of $\mathbf{y}$: this is T-dual to the statement that the $\mathbf{x}$ coordinates of the worldsheet satisfy Neumann boundary conditions when ending on a D3-brane:

$$0 = \partial_\tau \mathbf{x}|_{\partial\Sigma} \;\propto\; \partial_\sigma \mathbf{y}|_{\partial\Sigma}.$$

Therefore, the projection of the worldsheet into the Minkowski space is again a polygon specified by the momenta, with the order determined by the Chan-Paton ordering. As in the pure gluon amplitude, the components of the boundary ending on the D3-branes have Dirichlet boundary conditions in the radial direction (r). When the string ends on the D7-brane, however, it satisfies Neumann boundary conditions in the radial direction, and

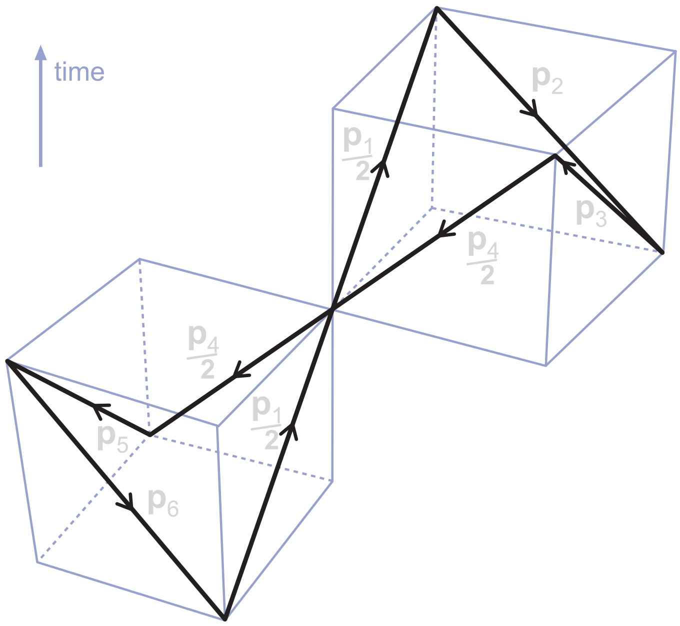


Figure 4: The polygon associated to orientifold-symmetric 6-gluon scattering.

can extend into the bulk. Since this component of the boundary lies at a fixed value of y, such an extension must fold back on itself (see figure 3).[15]

In summary, the inclusion of the quark-antiquark pair amplitude introduces a new kind of cusp to the light-like polygon Wilson loop, above which the worldsheet *may* end on a folded string. Next, we will see that it does.

5. Relating planar gluon scattering to quark scattering at strong coupling

AdS solutions for quark and gluon-quark amplitudes can be constructed from special gluon amplitudes. Specifically, consider a worldsheet ending on a light-like Wilson loop with a *self-crossing*, as follows.

For definiteness, we focus first on the $\bar{q}ggq$ (anti-quark, gluon, gluon, quark) amplitude. Consider the polygon associated to 6-gluon scattering with color-ordered momenta satisfying $\mathbf{p}_6 = \mathbf{p}_2$, $\mathbf{p}_5 = \mathbf{p}_3$. The conservation of momentum can now be written as

$$\frac{1}{2}\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \frac{1}{2}\mathbf{p}_4 = 0 \ . \tag{5.1}$$

The resulting polygon crosses itself at the midpoints of the lines associated with $\mathbf{p}_1$ and $\mathbf{p}_4$ (see figure 4). Let $\mathbf{y} = 0$ be the point of crossing. The polygon is mapped to itself by a symmetry which reflects through the crossing point, while simultaneously reversing the orientation of the gluon lines. Assuming there is a unique worldsheet which extremizes the area with these boundary conditions (see section 3.4), it too is mapped to itself by $R\Omega$,

[15] The possibility that the worldsheet could end on a folded string was independently recognized in [23] with, however, a contradicting conclusion.

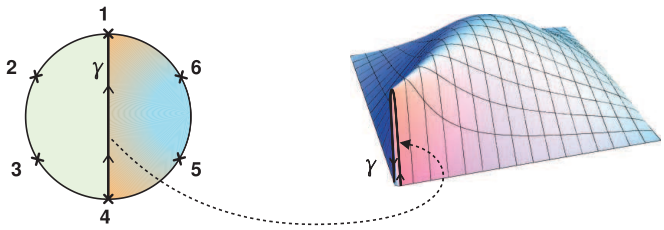


Figure 5: Instructions for cutting.

where

$$R: \qquad (\mathbf{y}, r) \to (-\mathbf{y}, r)\,, \tag{5.2}$$

extends the action on the momenta to the whole T-dual AdS space, and

$$\Omega: \qquad (\sigma, \tau) \ \to \ (-\sigma, \tau) \tag{5.3}$$

is the worldsheet parity; we have parameterized the disk by the upper half plane coordinates $(\sigma, \tau \geq 0)$.[16]

The six-gluon worldsheet is not yet known. Nevertheless, we can infer the following from the $R\Omega$ symmetry. Consider the two boundary points $\sigma_1 = 0$ and $\sigma_4 = \infty$ where the corresponding vertex operators are inserted. There is a curve on the worldsheet $\gamma = (0, \tau)$ which connects them (see figure 5), and which is made of points that are invariant under Ω. If we assume that the radial direction varies smoothly as we cross γ, then it follows from the $R\Omega$ symmetry that it satisfies Neumann boundary conditions there.[17]

To relate this picture to the quark amplitude, it is crucial that the string does not "open up" above the crossing point. That is, we require that the image of γ is a folded string at $\mathbf{y} = 0$. To see that this is indeed the case, note that modding out by $R\Omega$ is equivalent to placing an orientifold line that extends into the radial direction above that point ($\mathbf{y} = 0$). In that picture, our open string worldsheet is one-half of a closed string worldsheet with crosscap.

So far, we have argued that the extent of the folded string along the "orientifold line", l, can vary from 0 to ∞ in the configuration space. The following argument is evidence that $l > 0$. One half of the divergent piece of the self-crossing 6-gluon amplitude is smaller then the divergent piece in the 4-gluon amplitude. If the solution did not extend into the radial direction above the crossing point (l=0), it would satisfy the AM boundary conditions as well. If such a solution existed and, for the 4-gluon amplitude, were different from the AM solution, AM should have summed over it as well, and it would dominate over the contribution of their solution! But the contribution of their solution, by itself, agrees with the field theory prediction. Note that the AM four-gluon solution (explicitly given in [1]) does not satisfy Neumann boundary conditions in the radial direction at the D3 boundaries (D3-cusps).

[16] Note that $R\Omega$ acts in the same way on the $(\mathbf{x}, z)$-space (3.2).

[17] Note that from the worldsheet point of view, *at* the boundary, nether r or $z = 1/r$ are good coordinates and one had better use the coordinate $\phi = \log(z)$.

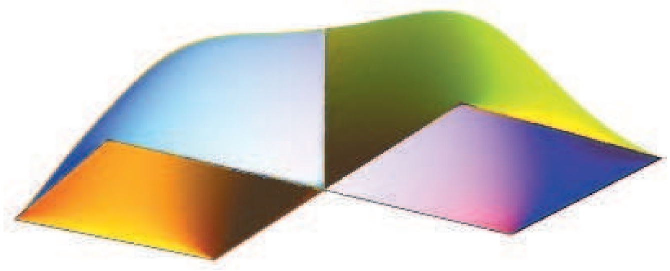


Figure 6: The 6-gluon worldsheet embedding passes through itself above the crossing. In doing so, it reverses orientation (indicated by color). Near the crossing, the embedding (in $\mathbb{R}^3$) is called a 'Whitney umbrella'.

If we cut the worldsheet along γ, each identical half is the desired worldsheet describing the $\bar{q}ggq$ scattering with quark momenta $\mathbf{k}_1 = \frac{1}{2}\mathbf{p}_1$ and $\mathbf{k}_4 = \frac{1}{2}\mathbf{p}_4$, and gluon momenta $\mathbf{p}_2, \mathbf{p}_3$. The configuration space for the $\bar{q}ggq$ worldsheet (made of all embeddings invariant under $R\Omega$) is a subspace of the one for the six-gluon worldsheet with crossings. Therefore, an extremum of the worldsheet path integral with 6-gluon boundary conditions that is $R\Omega$-symmetric, is in particular an extremum of the worldsheet path integral with the quark-scattering boundary conditions.[18]

By $R\Omega$-symmetry, the area is half the area of the six-gluon worldsheet. The value of the log of the $\bar{q}ggq$ amplitude ($\mathcal{S} = -\log(\mathcal{A})$) in the large-$\lambda$ planar limit, then, is half the value of the log of the six-gluon amplitude (see appendix 1 for notations):

$$\begin{aligned} -\mathcal{S}_{qgg\bar{q}}(k_1, k_2, k_3, k_4) &= -\frac{1}{2}\mathcal{S}_{6g}(2k_1, k_2, k_3, 2k_4, k_3, k_2) \\ &= -\frac{f(\lambda)}{8}\left[\ln^2\left(\frac{\mu^2}{-2s}\right) + \frac{1}{2}\ln^2\left(\frac{\mu^2}{-t}\right) - \frac{1}{2}\ln^2\left(\frac{4s}{-t}\right) + \mathrm{Li}_2\left(1+\frac{s}{t}\right) - \frac{9}{2}\zeta_2\right] \\ &\quad - \frac{g(\lambda)}{2}\left[\ln\left(\frac{\mu^2}{-2s}\right) + \frac{1}{2}\ln\left(\frac{\mu^2}{-t}\right)\right] - \frac{1}{4}h(\lambda)\,, \end{aligned} \tag{5.4}$$

where we have used the explicit expression for the finite part of the 6-gluon amplitude given in appendix A and $s = -(\mathbf{k}_1 + \mathbf{k}_2)^2$, $t = -(\mathbf{k}_2 + \mathbf{k}_3)^2$.[19] To get (5.4), one has to use the relation $t_i^{[n]} = t_{3-i}^{[6-n]}$. It follows from the $R\Omega$ symmetry where $i \to (3-i)$ implement the reflection and $n \to (6-n)$ implement the orientation change.[20]

The image of γ in the target space is the folded string. We expect the maximum value of r along γ to be of order $\sqrt{s_{q,\bar{q}}}$, where $s_{q,\bar{q}} = -(\mathbf{k}_q + \mathbf{k}_{\bar{q}})^2$ is the Mandelstam variable associated with the adjacent quark and anti-quark (s_{14} in the example above). The intuition is that it is advantageous in extremizing the area for the interior of the worldsheet to extend into the bulk of AdS, and this suspends the tip of the fold away from the boundary. Therefore the extension of the folded string into the radial direction

[18] Note that the reverse of this statement is not true. Here we rely on the assumption that there is a unique extremal surface in AdS corresponding to the ordering with crossing (see section 3.4).

[19] We remind the reader that in this paper we work with $(-+++)$ signature.

[20] Note that always $t_i^{[n]} = t_{n+i}^{[6-n]}$. Combining this relation with our symmetry gives $t_i^{[n]} = t_{3-n-i}^{[n]}$.

is expected to be of order $\sqrt{-\left(\sum_i \mathbf{k}_i\right)^2}$, where i runs over all momenta in the loop except the ones of the two adjacent quarks in question (so $\sum_i \mathbf{k}_i = -(\mathbf{k}_q + \mathbf{k}_{\bar{q}})$).

The six-gluon worldsheet that we have described has a singularity.[21] One can worry that such a development calls into question our classical description. The singular point is measure zero in the integral that produces the area; its only effect can come when we consider the variation of the action under small fluctuations around the extremum. The specific concern is that there could be a mode localized at the singularity which is a zeromode of the fluctuation matrix S''.[22] Since the singularity is stable, any zero mode that is localized at the singularity can only change its location but cannot smooth it out. One such mode is the mode that changes the height of the fold (l above). If we focus on the D7-cusp (along the lines of [7]), we find that l is not determined locally.[23] We expect the tension of the rest of the worldsheet provides a potential for it (proportional to $\sqrt{\lambda}$). The other dangerous modes change the location of the singularity in the four transverse directions (and are odd under $R\Omega$). Without the knowledge of the explicit solution we cannot determine their fate. Here we assume such zeromodes are absent.

Note that it is crucial for our picture that the crossing is along null lines. In [24], it is shown via AdS/CFT that the expectation value of an Euclidean Wilson loop with a self-crossing is equal to the sum of the expectation values of the two sub-loops. This result follows from the infinite area cost for the Euclidian string to extend near the boundary. Therefore, cutting the Euclidian solution at some $r = \epsilon$ away from the boundary divides it into two components (see figure 7 of [24]), each of which is the extremal worldsheet for one of the sub-loops.

In Minkowski space however, it does not cost area for the worldsheet to extend along a null surface - even if it is near the AdS boundary. Therefore, it needn't be true that the worldsheet pinches to the boundary at the crossing. Indeed, we have given evidence above that in our case of interest it does not. (In the next section we will further verify this in the dual field theory.) In such a case, slicing the worldsheet at some small distance from the boundary does *not* separate it into two disjoint Wilson loops (see figures 3 and 5).

As for the AM case, here, in the T-dual picture, we can ignore the dilaton unless the worldsheet extends into the region where it blows up. The dilaton admits its maximal value at the cutoff ($r_{UV} = \sqrt{\lambda}/z_{IR}$) and therefore, the presence of the D7 cusp does not invalidate that approximation.

5.1 Generalizations

By considering other symmetric gluon scattering amplitudes, we can construct the generic amplitude with quarks.

The next simplest case is two quarks and any number of gluons. Specifically, consider a scattering amplitude with two quarks with momenta $\mathbf{k}_1$, $\mathbf{k}_n$, and $n-2$ gluons with

[21] The Whitney umbrella is a *stable* singularity of a map from $\mathbb{R}^2 \to \mathbb{R}^3$; *stable* here means stable with respect to variations preserving the *k-jet* (for some k) of the singularity. Our solution seems to provide an interesting connection between jets in physics and jets in mathematics.

[22] Note that any non-zero eigenvalue of S'' is proportional to $\sqrt{\lambda}$.

[23] We thank Hong Liu for discussions of this point.

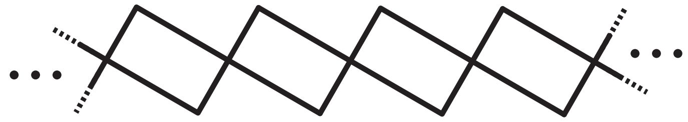


Figure 7: Prescription for the $q\bar{q}q\bar{q}$ amplitude.

momenta $\mathbf{k}_2, \ \ldots \ \mathbf{k}_{n-1}$. We will study an auxiliary $(2n-2)$-gluon scattering process with a $\mathbb{Z}_2$ symmetry $R\Omega$. Label the gluon momenta

$$\mathbf{p}_1 = 2\mathbf{k}_1\,, \qquad \mathbf{p}_n = 2\mathbf{k}_n\,, \qquad \mathbf{p}_{n+l} = \mathbf{p}_{n-l} = \mathbf{k}_{n-l}\,, \quad l = 1, \ \ldots \ n-2\ .$$

Momentum conservation of the $\bar{q}g^{n-2}q$ process ($\sum_{i=1}^n \mathbf{k}_i = 0$) means that the gluon momenta satisfy the condition

$$\frac{1}{2}\mathbf{p}_1 + \mathbf{p}_2 + \ \ldots \ + \mathbf{p}_{n-1} + \frac{1}{2}\mathbf{p}_n = 0, \tag{5.5}$$

which says that the $\mathbf{p}_1$ line and the $\mathbf{p}_n$ line crosses at their midpoints. The resulting polygon is $R\Omega$-symmetric and therefore, we expect the extremal worldsheet ending on it to be $R\Omega$-symmetric as well. The area for the $(2n-2)$-gluon amplitude is twice the amplitude for $\bar{q}g^{n-2}q$:

$$\mathcal{S}_{\bar{q}g^{n-2}q}(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{n-1}, \mathbf{k}_n) = \frac{1}{2}\mathcal{S}_{g^{2n-2}}(2\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{n-1}, 2\mathbf{k}_n, \mathbf{k}_{n-1}, \mathbf{k}_{n-2}, \ldots, \mathbf{k}_2)\ . \tag{5.6}$$

To construct an amplitude with more than one pair of quarks, proceed as follows. Draw the closed polygon associated with the color-ordered momenta of the quarks and gluons in question. The quarks must come in quark anti-quark adjacent pairs.[24] The quark lines in adjacent pairs meet at a D7-cusp like the one described in detail above. At each of these D7-cusps, append the image of the polygon under the $\mathbb{Z}_2$ symmetry which acts locally as described above. As soon as there is more than one pair of quarks, there will be infinitely many images (just as when mirrors face each other). We can regulate this by restricting the images to some finite volume V, and ending the graph in some way which breaks the symmetry but closes the color trace. Because this is a polygon in flat space, at large volume V, the contributions from the parts of the polygon near the boundary are negligible.

For example, we can construct the amplitude for $q\bar{q} \to q\bar{q}$ by considering the process with infinitely many gluons indicated in figure 7. This leads to the relation

$$\begin{aligned} &M_{\bar{q}q\bar{q}q}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \\ &\quad = \lim_{\ell\to\infty} \frac{1}{\ell} M_{g^{2\ell}}(\mathbf{k}_1, \underbrace{2\mathbf{k}_2, 2\mathbf{k}_1, 2\mathbf{k}_2, \ \ldots \ , 2\mathbf{k}_1}_{\ell-2 \text{ times}}, \mathbf{k}_2, \mathbf{k}_3, \underbrace{2\mathbf{k}_4, 2\mathbf{k}_3, 2\mathbf{k}_4, \ \ldots \ , 2\mathbf{k}_3}_{\ell-2 \text{ times}}, \mathbf{k}_4)\ . \end{aligned} \tag{5.7}$$

[24]The possible exception to this is the insertion of 7-7 strings between the quarks. According the the gauge/string dictionary, such states are dual to mesons. The corresponding amplitudes then describe 'form factors' for the scattering of quarks and gluons off of these mesons. We leave their study for future work.

6. No Sudakov form factors from D7 cusps

In the previous section we saw that the extremal area surfaces corresponding to quark scattering can be obtained from gluon ones with self-crossing. In this section we study the implications of this statement in field theory. We will first show that in one-loop gluon amplitudes, no new divergence arises from such a crossing. Given the structure of the BDS ansatz, this implies that no new divergence arises to all orders. Next, we give a pure field theory derivation of this prediction, by showing that the Sudakov form factors corresponding to D7 cusps are down by $1/N$ and are therefore absent in the strict large N limit. This supports the claim of §5 that on top of the crossing point, there is a folded string which extends in the radial direction, as drawn in figure 3.

6.1 One-loop gluon amplitude with crossing

As far as the one loop gluon integral is concerned, no new divergence arises when the loop made of the successive *null* external momenta develops a self-crossing. This can be shown by directly studying the $\mathcal{N} = 4$ six-gluon one-loop amplitude (which reduces to a scalar box integral, see e.g. [14] and references therein). An intuitive way to extract the contribution to the one-loop integral from the crossing is to study its "double loop" representation (2.8) [14, 15] reviewed in §2. In that representation, divergences arise when the two integrals run over adjacent legs of the polygon. As a preparation, consider first the contribution to (2.8) from the region of integration where both $\mathbf{y}$ and $\mathbf{y}'$ are near a cusp made of two successive momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. We parameterize $\mathbf{y}$ and $\mathbf{y}'$ as

$$\mathbf{y} = -\tau \mathbf{k}_1 \,, \qquad \mathbf{y}' = \tau' \mathbf{k}_2 \,, \tag{6.1}$$

where τ and τ' run from 0 to 1. The part of the integral (2.8) where $\mathbf{y}$ runs on the $\mathbf{k}_1$ segment and $\mathbf{y}'$ runs over the $\mathbf{k}_2$ segment is:

$$\begin{aligned} \mu^{2\epsilon} \int_{-\mathbf{k}_1}^{0} \int_{0}^{\mathbf{k}_2} \frac{d\mathbf{y}\cdot d\mathbf{y}'}{[-(\mathbf{y}-\mathbf{y}')^2]^{1+\epsilon}} &= \frac{1}{2}\left(\frac{\mu^2}{-2\mathbf{k}_1\cdot\mathbf{k}_2}\right)^{\epsilon} \int_1^0 \frac{d\tau}{\tau^{1+\epsilon}} \int_0^1 \frac{d\tau'}{\tau'^{1+\epsilon}} \\ &= -\frac{1}{2}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{1,2}}\right)^{\epsilon} , \end{aligned} \tag{6.2}$$

where it was crucial that $\epsilon < 0$ for (6.2) to converge.

Now consider a loop Π_∞ made of successive *null* segments that has a single crossing. It can be thought of as two loops $\Pi_{1,2}$ touching at a cusp point with completing angles and opposite orientations, $\Pi_\infty = \Pi_1 \cup \Pi_2$. The double loop integral (2.8) now takes the form

$$\begin{aligned} M_1^{(n)} &= \frac{1}{2}\mu^{2\epsilon} \oint_{\Pi_1\cup\Pi_2} \oint_{\Pi_1\cup\Pi_2} \frac{d\mathbf{y}\cdot d\mathbf{y}'}{[-(\mathbf{y}-\mathbf{y}')^2]^{1+\epsilon}} \\ &= \frac{1}{2}\mu^{2\epsilon}\left(\oint_{\Pi_1}\oint_{\Pi_1} + \oint_{\Pi_2}\oint_{\Pi_2} + 2\oint_{\Pi_1}\oint_{\Pi_2}\right) \frac{d\mathbf{y}\cdot d\mathbf{y}'}{[-(\mathbf{y}-\mathbf{y}')^2]^{1+\epsilon}} \,. \end{aligned} \tag{6.3}$$

The first two double loop integrals in the last line of (6.3) are just (2.8) evaluated on the loops Π_1 and Π_2. If that were the whole answer one would conclude that the intersection

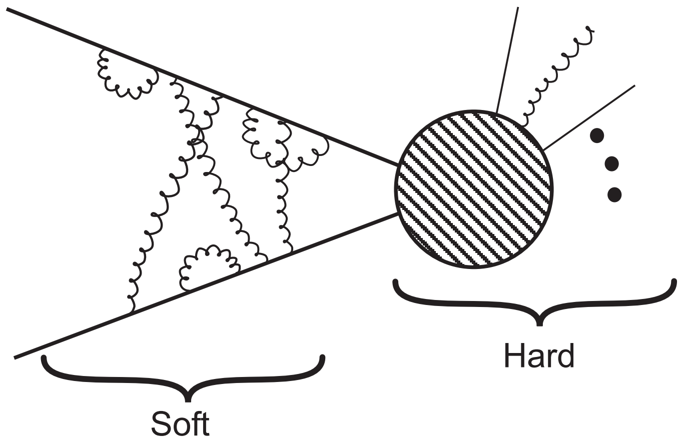


Figure 8: An example of the soft gluon exchange diagrams that resum into the Sudakov form factor. The curled lines are gluons, where the external line from which the soft gluons are emitted represent any colored particles.

gave double the divergent contribution (6.2) (which after expanding in powers of ϵ are the one-loop contribution to the Sudakov form factors). However, the double cross-loop integrals in the last line of (6.3) exactly cancel these divergences. To see this cancellation, we simply include the cross-terms in (6.3) by allowing τ and τ' in (6.2) to proceed beyond the zero:

$$\int_1^0 \frac{d\tau}{\tau^{1+\epsilon}} \int_0^1 \frac{d\tau'}{\tau'^{1+\epsilon}} \quad \rightarrow \quad \int_1^{-1} \frac{d\tau}{\tau^{1+\epsilon}} \int_{-1}^1 \frac{d\tau'}{\tau'^{1+\epsilon}} = \left[\frac{1-(-1)^{-\epsilon}}{\epsilon}\right]^2 \tag{6.4}$$

which is finite as $\epsilon \to 0$. In some abused language, it is the presence of these cross terms between the two loops that causes the worldsheet to extend into the radial direction above the crossing point.

6.2 Sudakov and the $1/N$ expansion

A colored particle participating in a scattering process is very unlikely not to emit soft gluons. In the perturbative calculation of exclusive amplitudes, this statement manifests itself in the form of an IR divergent exponential suppression factor. In physical quantities, these divergences are replaced by and determine important dependence on the definition of the observable (detector sensitivity, cone angles) [4]. These divergences can be resummed into a 'Sudakov factor' of the form [25−27, 12, 28, 2] (and see [29] for a recent discussion in this context),

$$\mathcal{A} \sim e^{-h(\lambda)\log^2(\mu_{IR})-h'(\lambda)\log(\mu_{IR})}\,, \tag{6.5}$$

where h, h' are some functions of the coupling. The form of the soft gluon exchange diagrams that contribute to the Sudakov form factor is shown in figure 8.

For massless external particles, the Sudakov form factor looks like

$$e^{-\frac{f(\lambda)}{4}\log^2(\mu)} \tag{6.6}$$

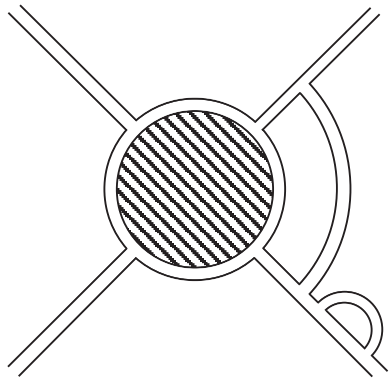


Figure 9: An example of a planar soft 4-gluon exchange diagram that contribute to the corresponding Sudakov form factor.

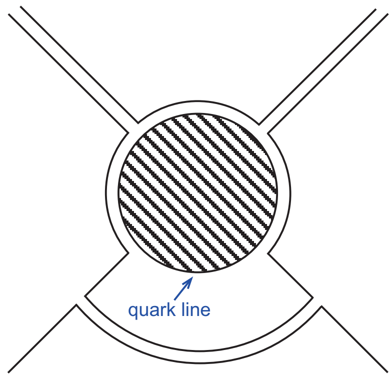


Figure 10: An example of a quark anti-quark to 2-gluon diagram where the quark and anti-quark exchange soft gluon. It has an annulus topology and is therefore down by $1/N$ with respect to the same diagram without the soft gluon exchange.

as a function of the IR cutoff μ [25, 26, 12, 28, 29] (for a review see [27]), where $f(\lambda)$ is the cusp anomalous dimension. This factor appears in front of the *whole* amplitude, it gives the leading IR behavior when we consider the exclusive scattering of colored particles, and it is an important ingredient in the computation of amplitudes [5].

In a planar diagram contributing to gluon scattering, the left index line from each gluon ends at the right index line of the next. For each consecutive pair we get a factor of the form (6.5), with the function h given by $f/4$. An example of such a planar soft gluon exchange diagram is drawn in figure 9 in the double line notation.

Imagine now that we replace two successive gluons by quark anti-quark pair. As can be seen in figure 10, a gluon exchange between the quark and anti-quark results in a hole in the worldsheet and is therefore down by $1/N$. This does not mean that there is no Sudakov form factor from soft gluon exchange between the quark and the anti-quark. All it means is that these diagrams combine with the non-planar diagrams so that the coefficient in front of the $\log^2(\mu)$ in the Sudakov form factor scales as $1/N$ and disappears if we take the large N limit while holding the IR cutoff (μ) fixed. Note that any external quark does lead to a Sudakov form factor that is not down by $1/N$. It results from exchanging soft gluons with

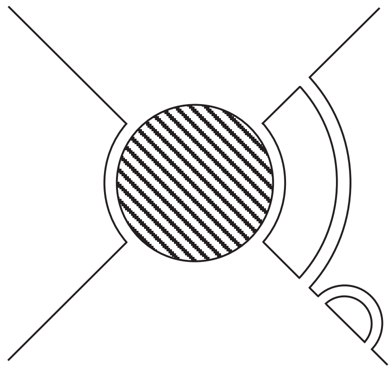
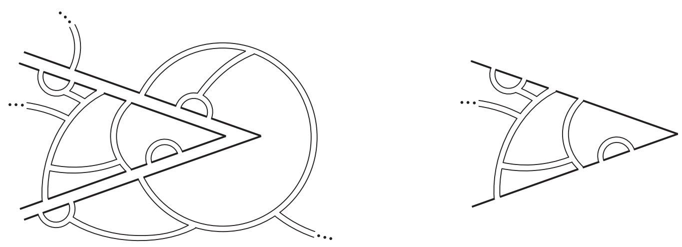


Figure 11: An example of a planar quark anti-quark to quark anti-quark diagram where successive quark anti-quark exchange a soft gluon. It contribute to the corresponding Sudakov form factor and is not down by $1/N$.



a. b.

Figure 12: An example of planar diagram that contributes to the flux tube extended from Wilson line with cusp. **a.** Wilson loop in the adjoint has two flux tubes. **b.** Wilson loop in the fundamental has one flux tube.

the next hard quark or gluon *along the same color line*, as in figure 11. This divergence, however, corresponds to a D3-cusp in the dual string picture.

A limit of the AM construction is the statement that the Sudakov double logs can be computed by replacing the hard partons by Wilson loops [26, 30]. The factor in front of the double logarithm ($f(\lambda)$, the cusp anomalous dimension) for a Wilson line in the fundamental representation is half the one for a Wilson line in the adjoint representation. That factor of half is exactly the manifestation of what we have just seen. As was recently explained in [29], the coefficient $f(\lambda)$ in the Sudakov form factor is the tension of the flux tube that is stretched from the colored particle. The fact that for a Wilson line in the adjoint representation this factor is twice the one for a Wilson line in the fundamental representation can be interpreted as saying that there are two flux tubes stretching out of a particle in the adjoint representation, one to the 'right' and one to the 'left' with respect to the planar structure of the amplitude (see figure 12). In the Wilson line picture, they are folded on top of each other and are equivalent to a single flux tube with doubled tension.

7. Comparison with perturbative results

7.1 Non-BDS for quark amplitudes

As was checked up to 4-loops (see [31] for 5-loops ansatz) and conjectured by BDS [2], the planar MHV one-loop gluon amplitude in $N = 4$ SYM exponentiates (see section 2). One might naively expect that a similar exponentiation holds for planar HMV one-loop massless quark amplitudes in $\mathcal{N} = 2$ SYM (in the probe approximation). Such a relation would make our conjecture easy to check. However, this cannot be the case. In the pure gluon amplitude, one would picturesquely expect that when expanding the exponent of the one-loop amplitude, many such one-loop amplitudes will "tile" together to form the string worldsheet. In the t' Hooft large N expansion of the quark-gluon amplitude, the quark propagators must sit on the boundary of the worldsheet (see figure 11). An exponentiation of the one-loop quark-gluon amplitude would have too many quark propagators to form a continuous worldsheet where quark propagators lay on its boundary only.[25]

Assuming our prediction is true, there is another more technical argument we can make for the non-exponentiation of the planar one-loop quark amplitude. That is, the six gluon one-loop amplitude contains a dilogarithm term ($\mathrm{Li}_2(1 + s/t)$ in (5.4)), whereas the one-loop amplitude with four massless external legs cannot have dilogarithm. This can be seen by noting that such an amplitude can be written as a linear combination of scalar one-loop amplitudes with coefficients that are at most rational functions of the Mandelstam variables. The scalar one-loop amplitudes that can appear are the scalar box amplitude with four massless external legs, scalar triangle amplitude with one massive and two massless external legs, and the scalar 'diangle' amplitude, i.e. the scalar one-loop amplitude with two propagators and two massive external legs. Since none of these scalar one-loop amplitudes gives a dilogarithm, the one-loop two quarks to two gluons amplitude (times the tree-level one) cannot match the corresponding six gluon one-loop amplitude.

Next, we would like to check if our conjectural relation to the gluon amplitude could hold order by order in perturbation theory. What could extrapolation to small λ of the relation between the two quarks - two gluon amplitude and the six gluon amplitude mean? At least naively, it means that order by order in perturbation theory, the quark-gluon amplitude is the square root of the six gluon one (see figure 13). That is obviously wrong as can be seen for example by noting again that the one-loop quark gluon amplitude does not contain a dilogarithm, whereas the six gluon one at one and two loops amplitude does. It would be nice to have a prediction for the all-orders amplitudes in the $\mathcal{N} = 2$ theory.

7.2 Regge behavior of quark scattering amplitudes in $\mathcal{N} = 2$ SYM

In the previous subsection we have learned that the planar one-loop massless quark amplitudes in $\mathcal{N} = 2$ SYM (in the probe approximation) does not exponentiate. We have also seen that naive extrapolation of our strong coupling conjectural relation to one-loop would contradict our prediction. A comparison of our strong coupling prediction with (weak coupling) perturbative results may still be possible in some special limits, where an all-loop

[25] We thank Howard Schnitzer for raising that point.

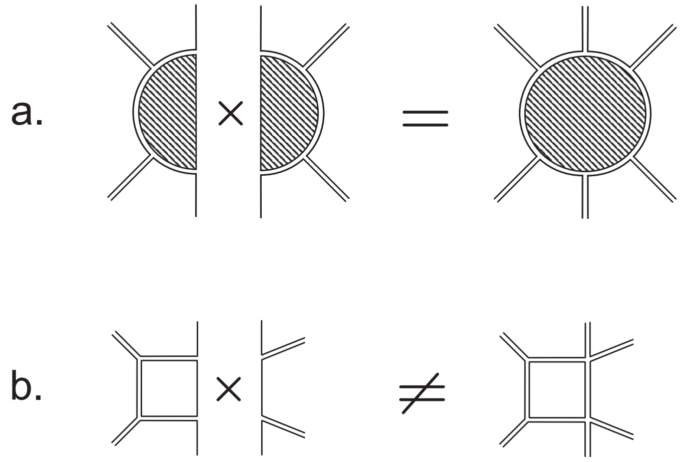


Figure 13: a) Diagrammatic form of our strong coupling conjecture. **b)** One of the diagrams that would contribute in a naive extrapolation of our conjecture to one-loop.

perturbative computation is in reach. Such a limit is the Regge limit. In this limit it is believed that, by summing over ladder diagrams [32], one can obtain an all-loops planar result.

Therefore, in this section we investigate the Regge limit of the on-shell planar two quarks $\to$ two gluons amplitude in strongly coupled $\mathcal{N} = 2$ SYM, using our explicit prediction (5.4). The Regge limit is the limit where the center of mass energy squared is taken to infinity (and is therefore timelike) with fixed spacelike momentum transfer.

In the notation of section 5, the Mandelstam variables are $s = -(\mathbf{k}_1 + \mathbf{k}_2)^2$, $t = -(\mathbf{k}_2 + \mathbf{k}_3)^2$ and $u = -(\mathbf{k}_2 + \mathbf{k}_4)^2$, where $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ are the momentum of $(q_1, g_2, g_3, \bar{q}_4)$ correspondingly. There are four different Regge limits we can take here:[26]

(1) q_1 and g_3 are incoming and g_2, $\bar{q}_4$ are outgoing. In that case center of mass energy squared is $u \to \infty$, whereas the fixed spacelike momentum transfer is $s < 0$ (therefore $t \to -\infty$). That is the convention in which u, s and t are usually used. Here, to keep things short, we will use the same expressions for u, s and t in terms of the momenta in the three other cases as well.

(2) q_1 and g_3 are incoming and g_2, $\bar{q}_4$ are outgoing, $u \to \infty$ with fixed $t < 0$.

(3) q_1 and g_2 are incoming and g_3, $\bar{q}_4$ are outgoing, $s \to \infty$ with fixed $t < 0$.

(4) q_1 and $\bar{q}_4$ are incoming and g_2, g_3 are outgoing, $t \to \infty$ with fixed $s < 0$.

An amplitude is said to have Regge behavior if in the Regge limit it approaches

$$\mathcal{A}_{qgg\bar{q}}(a, b) = \beta(b) \left(\frac{a}{-b} \right)^{\alpha(b)} \left[1 + \mathcal{O}(|b|/a) \right] , \tag{7.1}$$

where $a \in \{u, s, t\}$ is the center of mass energy, $b \in \{u, s, t\}$ is the fixed spacelike momentum transfer, $\alpha(b)$ is the Regge trajectory and $\beta(b)$ is the Regge residue.

[26]The other two possible limits will be discussed below.

In [14] and [33], assuming the BDS ansatz, the planar four-gluon amplitude in $\mathcal{N}=4$ SYM was found to possess Regge behavior, which was further analyzed. In [14], the Regge limit of the *in* $\to$ *in* $\to$ *out* $\to$ *out* amplitude was analyzed using [2]. It was found that the amplitude is Regge exact.[27] In [33], based on the BDS ansatz and the strong coupling prediction [1], leading Regge trajectory was found together with an infinite number of daughter trajectories and analyzed in the *in* $\to$ *out* $\to$ *in* $\to$ *out* amplitude. Next we will see that the two-quark — two-gluon amplitude has Regge behavior in four different channels (but is never Regge exact).

It follows from (7.1) that in the Regge limit, the Regge trajectory is given by

$$\alpha(b) = -\frac{\partial}{\partial \ln a}\mathcal{S}_{\bar{q}ggq}(a,b)\,, \tag{7.2}$$

(recall that $\mathcal{S} \equiv -\ln \mathcal{A}$). Note that an amplitude is dominated by a Regge trajectory in the Regge limit only if the leading term on the right hand side of (7.2) is a-independent. To analyze our prediction for the amplitude (5.4) in the Regge limit, we need the asymptotic expansion of the term $\mathrm{Li}_2\,(1+x)$ in the limits $|x| \to \infty$ and $|x| \to 0$. For that aim we use the relations

$$\begin{aligned}
\mathrm{Li}_2(1+x) + \mathrm{Li}_2(1+x^{-1}) &= -\frac{1}{2}\ln^2(-x)\,,\\
\mathrm{Li}_2(-x) + \mathrm{Li}_2(1+x) &= \frac{1}{6}\pi^2 - \ln(-x)\ln(1+x)\,,\\
\mathrm{Li}_2(x) &= \sum_{k=1}^{\infty}\frac{x^k}{k^2}\;.
\end{aligned} \tag{7.3}$$

For $|x| \to \infty$ we combine these as

$$\mathrm{Li}_2\,(1+x) = -\frac{1}{2}\ln^2(-x) + \ln\left(-x^{-1}\right)\ln\left(1+x^{-1}\right) - \frac{1}{6}\pi^2 + \sum_{k=1}^{\infty}\frac{(-x)^{-k}}{k^2} \tag{7.4}$$

whereas, for $|x| \to 0$

$$\mathrm{Li}_2\,(1+x) = \frac{1}{6}\pi^2 - \ln(-x)\ln(1+x) - \sum_{k=1}^{\infty}\frac{(-x)^k}{k^2}\;. \tag{7.5}$$

By plugging these equations into (5.4) we see that in each of the Regge limits, the $\ln^2(a)$ term exactly cancels. We list below the results for $\alpha_3(s)$ and $\alpha_4(s)$. The two other Regge trajectories and the four finite parts can be easily computed, however we will not need them here. We find that

$$\begin{aligned}
\alpha_3(t) &= -\frac{1}{4}f(\lambda)\ln\left(\frac{t}{\mu^2}\right) + \frac{1}{2}g(\lambda) + \frac{1}{4\epsilon}f^{(-1)}(\lambda) + [\text{dressing contribution}]\\
\alpha_4(s) &= -\frac{1}{8}f(\lambda)\ln\left(\frac{4s}{\mu^2}\right) + \frac{1}{4}g(\lambda) + \frac{1}{8\epsilon}f^{(-1)}(\lambda) + [\text{dressing contribution}]\,,
\end{aligned} \tag{7.6}$$

[27] That is, for any values of t and s it can be written in the Regge form (7.1), with no subleading corrections ($\mathcal{O}(|t|/s)$ in (7.1)).

where $\lambda\frac{\partial}{\partial\lambda}f^{(-1)}(\lambda) = f(\lambda)$ and the [dressing contribution] is the contribution from the helicity-dependent kinematic factors that multiply the exponential.

First, we claim that the fact that our prediction leads to Regge trajectories in the Regge limit is a non-trivial consistency check. The reason is the following. These amplitudes are IR divergent and need an IR regulator. One such IR regulator is implemented by introducing a mass-gap [32]. In the Regge limit one takes the UV limit where the center of mass energy diverges (with fixed 4-momentum transfer), keeping the IR regulator fixed. There is a general proof that renormalizable non-abelian gauge theories with a mass gap, lead to Regge trajectories for the elementary fields of the theory [34].[28] In the probe approximation, the $\mathcal{N}=2$ theory at hand is conformal and therefore renormalizable. The existence of these trajectories should be independent of the choice of IR regulator, although the precise behavior of the trajectory itself ($\alpha(b)$), in the IR region, will depend on the details of the IR regulator chosen for the massless fields.[29]

As stated in the beginning of this section, in the Regge limit we can actually compare terms in (7.6) with a perturbative all-loop computation. These are the coefficients of the leading log contributions to the Regge trajectories, $-\frac{1}{4}f(\lambda)$ and $-\frac{1}{8}f(\lambda)$ in α_3 and α_4 correspondingly. In the perturbative region these become $-\frac{\lambda}{8\pi^2}$ and $-\frac{\lambda}{16\pi^2}$. In planar $\mathcal{N}=4$ SYM, the leading log coefficient of the Regge trajectory is $-\frac{1}{4}f(\lambda)$. In the perturbative regime, it is captured by a specific infinite subclass of diagrams — the ladder graphs [32]. A generic $\mathcal{N}=4$ ladder diagram that contributes to the four gluon Regge trajectory in the s-channel is drawn in figure 14.a. For any given ladder diagram, if we replace two of the external gluons on the left with quark anti-quark pair, as drawn in figure 14.b, then only the first 'rung' of the ladder is changed and the rest of the ladder remains the same. After summing the leading logs of all these ladder diagrams, such a change can only affect the dressing factor in front of the exponent. We therefore conclude that in case 3) of the quark-gluon Regge limit considered above, the leading log contributions to the Regge trajectory must be the same as the one in the $\mathcal{N}=4$ gluon Regge trajectory — in agreement with what we have found (7.6). If instead of case 3), we consider case 4), then each of the ladder diagram looks as in figure 14.c. In that case the basic block in the ladder is changed. We conjecture that these still sum to give the leading log contributions to the corresponding Regge trajectory but leave the check for future work. If true, following our results for α_4, we expect it to be one-half the one for gluons ($-\frac{\lambda}{8\pi^2}$).

There are other two Regge limits we can take that are not listed above. These are the limits where $s \to \pm\infty$ with $u < 0$ fixed. In these limits the fixed momentum transfer is not between two color-adjacent partons. Therefore, the ladder diagrams that perturbatively lead to the Regge trajectory are not planar. We then expect the coefficent in front of the corresponding Regge term (that dominates the amplitude at finite N, $|s| \to \infty$) to scale as $1/N$ and to be absent in the strict planar limit. That is, the amplitude takes the rough form

$$\mathcal{A}_{\text{other two channels}} \sim \frac{\gamma(b)}{N}\left(\frac{a}{-b}\right)^{\alpha(b)} + \eta(b)e^{-\chi(\lambda)\ln^2(-a/b)}\,, \tag{7.7}$$

[28] Moreover, in such theories $\alpha(0)$ is the elementary spin (that is, 1 for vectors and 1/2 for fermions).

[29] We thank Howard Schnitzer for explaining the related facts to us.

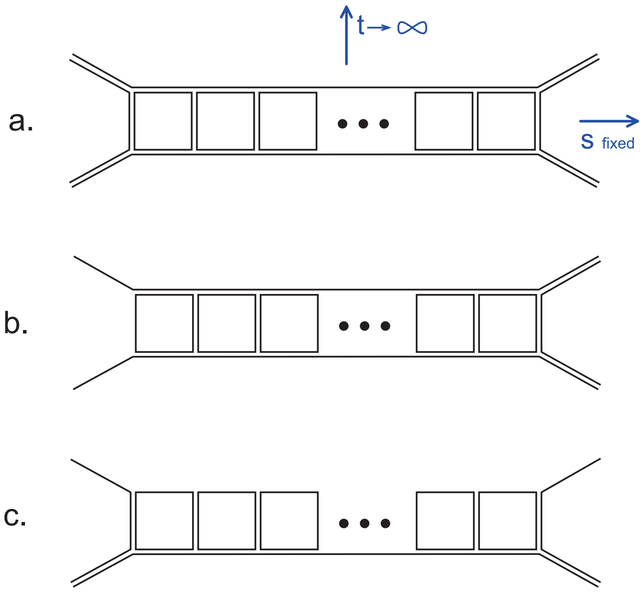


Figure 14: a) A generic ladder diagram that contribute to the leading *log* in the $\mathcal{N} = 4$ planar Regge trajectory. **b, c)** Generic ladder diagrams in quark gluon planad amplitude.

where the functions $\gamma(b)$ and $\eta(b)$ scales as N^0 in the large N limit. Indeed, only in these two cases, our prediction (5.4) does not lead to Regge behavior in the Regge limit.

8. Brief conclusions

In this paper, we generalized the prescription of Alday and Maldacena [1][30] to include fields in the fundamental representation. A direct solution of the resulting extremal-area problem is hard. We circumvented this difficulty by relating its solution to an auxiliary $\mathcal{N} = 4$ amplitude known from [2].

The resulting answer passed several checks. The IR divergent parts reproduce the Sudakov factors of a large-N theory with fundamentals. The answer has planar Regge behavior in the channels where it should. In one channel (See figure 14.b), the leading log of the trajectory matches that of the $\mathcal{N} = 4$ theory as it should.

Our main result, then, is a relationship between amplitudes in two field theories. It would be nice to have some direct understanding of the origin of this relation.

Note added. After this paper was published, [36] suggested that the BDS ansatz should be corrected for large numbers of gluons. Although some of our consistency checks assume the BDS ansatz for the six gluons amplitude, the main point of our paper, relating scattering amplitudes with quarks to pure gluon amplitudes, is independent of that ansatz.

Acknowledgments

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[30]Other related papers include [35].

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A. The one-loop finite remainder, $F_n^{(1)}$

The one-loop finite remainder, $F_n^{(1)}$ appearing in (2.7), was evaluated in [13]. It is given by (at $\epsilon = 0$):

$$F_n^{(1)} = \frac{1}{2}\sum_{i=1}^{n}\left[D_{n,i} + L_{n,i} - \sum_{r=2}^{[n/2]-1} \ln\left(\frac{t_i^{[r]}}{t_i^{[r+1]}}\right)\ln\left(\frac{t_{i+1}^{[r]}}{t_i^{[r+1]}}\right) + \frac{3}{2}\zeta_2\right] \tag{A.1}$$

where $[x]$ is the greatest integer less than or equal to x. Here $t_i^{[r]} = -(k_i + \cdots + k_{i+r-1})^2$ are the momentum invariants, so that $t_i^{[1]} = 0$ and $t_i^{[2]} = s_{i,i+1}$. (All indices are understood to be mod n.) The form of $D_{n,i}$ and $L_{n,i}$ depends upon whether n is odd or even. For $n = 2m+1$,

$$\begin{aligned} D_{2m+1,i} &= -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right), \\ L_{2m+1,i} &= -\frac{1}{2}\ln\left(\frac{t_i^{[m]}}{t_{i+m+1}^{[m]}}\right)\ln\left(\frac{t_{i+1}^{[m]}}{t_{i+m}^{[m]}}\right), \end{aligned} \tag{A.2}$$

whereas for $n = 2m$,

$$\begin{aligned} D_{2m,i} &= -\sum_{r=2}^{m-2} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right) - \frac{1}{2}\mathrm{Li}_2\left(1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}}\right), \\ L_{2m,i} &= -\frac{1}{4}\ln\left(\frac{t_i^{[m]}}{t_{i+m+1}^{[m]}}\right)\ln\left(\frac{t_{i+1}^{[m]}}{t_{i+m}^{[m]}}\right). \end{aligned} \tag{A.3}$$

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